Relationship between Conditional Diagnosability and 2-extra Connectivity of Symmetric Graphs*

Rong-Xia Hao[a] Zeng-Xian Tian[a] and Jun-Ming Xu[b†‡]

[a]Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, China

[b]School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, China

Abstract

The conditional diagnosability and the 2-extra connectivity are two important parameters to measure ability of diagnosing faulty processors and fault-tolerance in a multiprocessor system. The *conditional diagnosability* $t_c(G)$ of G is the maximum number t for which G is conditionally t-diagnosable under the comparison model, while the 2-*extra connectivity* $\kappa_2(G)$ of a graph G is the minimum number k for which there is a vertex-cut F with $|F|=k$ such that every component of $G-F$ has at least 3 vertices. A quite natural problem is what is the relationship between the maximum and the minimum problem? This paper partially answer this problem by proving $t_c(G)=\kappa_2(G)$ for a regular graph G with some acceptable conditions. As applications, the conditional diagnosability and the 2-extra connectivity are determined for some well-known classes of vertex-transitive graphs, including, star graphs, (n,k)-star graphs, alternating group networks, (n,k)-arrangement graphs, alternating group graphs, Cayley graphs obtained from transposition generating trees, bubble-sort graphs, k-ary n-cube networks and dual-cubes. Furthermore, many known results about these networks are obtained directly.

Keywords conditional diagnosability; comparison model; extra connectivity; symmetric graph; Cayley graph; max-min problem

1 Introduction

Throughout this paper, unless otherwise specified, a graph $G=(V,E)$ is always assumed to be a simple and connected graph, where $V=V(G)$ is the vertex-set and

*The work was supported by NNSF of China No. 11371052, 11171020, 11271012, 61272008.

†Corresponding author: xujm@ustc.edu.cn (J.-M. Xu)

‡Email addresses: rxhao@bjtu.edu.cn (R.-X. Hao), 14121550@bjtu.edu.cn (Z.-X. Tian).

$E = E(G)$ is the edge-set of G. We follow [41] for terminologies and notations not defined here.

Two distinct vertices x and y in G is adjacent if $xy \in E(G)$ and non-adjacent otherwise. If $xy \in E(G)$, then y (resp. x) is a neighbor of x (resp. y). The neighbor-set of x is denoted by $N_G(x) = \{y \in V(G) : \ xy \in E(G)\}$. For a subset $X \subset V(G)$, the notation $G - X$ denotes the subgraph obtained from G by deleting all vertices in X and all edges incident with vertices in X, and let $\overline{X} = V(G - X)$.

It is well known that a topological structure of an interconnection network N can be modeled by a graph $G = (V, E)$, where V represents the set of components such as processors and E represents the set of communication links in N (see a text-book by Xu [42]). Faults of some processors and/or communication lines in a large-scale system are inevitable. People are concerned with how to diagnose faults and to determine fault tolerance of the system.

A vertex in a graph G is called a *fault-vertex* if it corresponds a faulty processor in the interconnection network N when it is modeled by G. A subset $F \subseteq V(G)$ is called a *fault-set* if every vertex in F is a faulty vertex in G, and is *fault-free* if it contains no faulty vertex in G. A fault-set F is called a *conditional fault-set* if $N_G(x) \nsubseteq F$ for any $x \in \overline{F}$. The pair (F_1, F_2) is called a *conditional fault-pair* if both F_1 and F_2 are conditional fault-sets.

The ability to identify all faulty processors in a multiprocessor system is known as system-level diagnosis. Several system-level self-diagnosis models have been proposed for a long time. One of the most important models is the *comparison diagnosis model*, shortly *comparison model*. Throughout this paper, we only consider the comparison model.

The comparison model was proposed by Malek and Maeng [35, 36]. A node can send a message to any two of its neighbors which then send replies back to the node. On receipt of these two replies, the node compares them and proclaims that at least one of the two neighbors is faulty if the replies are different or that both neighbors are fault-free if the replies are identical. However, if the node itself is faulty then no reliance can be placed on this proclamation. According as that the two outputs are identical or different, one gets the outcome to 0 or 1. The collection of all comparison results forms a syndrome, denoted by σ.

A subset $F \subseteq V(G)$ is a *compatible fault-set* of a syndrome σ or σ is *compatible with* F, if σ can arise from the circumstance that F is a fault-set and $\overline{F}$ is fault-free. Let $\sigma_F = \{\sigma : \ \sigma \ is$ compatible with $F\}$. A pair (F_1, F_2) of two distinct compatible fault-sets is *distinguishable* if and only if $\sigma_{F_1} \cap \sigma_{F_2} = \emptyset$, and (F_1, F_2) is

indistinguishable otherwise.

For a positive integer t, a graph G is *conditionally t-diagnosable* if every syndrome σ has a unique conditional compatible fault-set F with $|F| \leqslant t$. The *conditional diagnosability* of G under the comparison model, denoted by $t_c(G)$ and proposed by Lai et al. [30], is the maximum number t for which G is conditionally t-diagnosable. The conditional diagnosability better reflects the self-diagnostic capability of networks under more practical assumptions, and has received much attention in recent years. The diagnosability of many interconnection networks have been determined, see, for example, [2, 3, 14–16, 20, 29, 40]. A survey on this field, from the earliest theoretical models to new promising applications, is referred to Duarte *et al.* [13].

A subset $X \subset V(G)$ is called a *vertex-cut* if $G - X$ is disconnected. A vertex-cut X is called a k-cut if $|X| = k$. The *connectivity* $\kappa(G)$ of G is defined as the minimum number k for which G has a k-cut.

Fault-tolerance or reliability of a large-scale parallel system is often measured by the connectivity $\kappa(G)$ of a corresponding graph G. However, the connectivity has an obvious deficiency because it tacitly assumes that all vertices adjacent to the same vertex of G could fail at the same time, but that is almost impossible in practical network applications. To compensate for this shortcoming, Fàbrega and Fiol [17] proposed the concept of the extra connectivity.

For a non-negative positive integer h, a vertex-cut X is called an *R_h-vertex-cut* if every component of $G - X$ has at least $h + 1$ vertices. For an arbitrary graph G, R_h-vertex-cuts do not always exist for some h. For example, a cycle of order 5 contains no R_2-vertex-cut. A graph G is called an R_h-graph if it contains at least one R_h-vertex-cut. For an R_h-graph G, the *h-extra connectivity* of G, denoted by $\kappa_h(G)$, is defined as the minimum number k for which G contains an R_h-vertex-cut F with $|F| = k$. Clearly, $\kappa_0(G) = \kappa(G)$. Thus, the h-extra connectivity is a generalization of the classical connectivity and can provide more accurate measures regarding the fault-tolerance or reliability of a large-scale parallel system and therefore, it has received much attention (see Xu [42] for details). We are interested in the 2-extra connectivity of a graph in this paper.

Clearly, for a graph G there are two problems here, one is the maximizing problem – conditional diagnosability $t_c(G)$, and another is the minimizing problem – the 2-extra connectivity $\kappa_2(G)$. A quite natural problem is what is the relationship between the maximum and the minimum problems? In the current literature, people are still determining these two problems independently for some classes of graphs, such as alternating group network [47], alternating group graph [21,45,50], the 3-ary

n-cube network [46].

In this paper, we reveal the relationships between the conditional diagnosability $t_c(G)$ and the 2-extra connectivity $\kappa_2(G)$ of a regular graph G with some acceptable conditions by establishing $t_c(G) = \kappa_2(G)$. As applications of our result, we consider some more general well-known classes of vertex-transitive graphs, such as star graphs, (n,k)-star graphs, alternating group networks, (n,k)-arrangement graphs, alternating group graphs, Cayley graphs obtained from transposition generating trees, bubble-sort graphs, k-ary n-cube networks and dual-cubes, and obtain the conditional diagnosability under the comparison model and the 2-extra connectivity of these graphs, which contain all known results on these graphs.

The rest of the paper is organized as follows. Section 2 first recalls some necessary notations and lemmas, then establishes the relationship between the conditional diagnosability and the 2-extra connectivity of regular graphs with some conditions. As applications of our main result, Section 3 determines the conditional diagnosability and the 2-extra connectivity for some well-known classes of vertex-transitive graphs.

2 Main results

We first recall some terminologies and notation used in this paper. Let $G = (V, E)$ be a graph, where $V = V(G)$, $E = E(G)$ and $|V(G)|$ is the order of G.

A sequence $(x_1, \cdots, x_n)$ of n ($\geqslant 3$) distinct vertices with $x_i x_{i+1} \in E(G)$ for each $i = 1, \cdots, n-1$ is called an n-path, denoted by P_n, if $x_1 x_n \notin E(G)$, and called an n-cycle, denoted by C_n, if $x_1 x_n \in E(G)$. A cycle C in G is *chordless* if any two non-adjacent vertices of C are non-adjacent in G.

For $X \subset V(G)$, let $N_G(X) = (\cup_{x\in X}\ N_G(x)) \setminus X$. For simplicity of writing, in case of no confusion from the context, we write $N(x)$ for $N_G(x)$; moreover, if X is a subgraph of G, we write $N(X)$ for $N_G(V(X))$ in this paper. For two non-adjacent vertices x and y in G, let $\ell(x,y) = |N(x) \cap N(y)|$, and let $\ell(G) = \max\{\ell(x,y) :\ x, y \in V(G)$ and $xy \notin E(G)\}$.

The degree $d(x)$ of a vertex x is the number of neighbors of x, i.e., $d(x) = |N(x)|$. The minimum degree $\delta(G) = \min\{d(x) :\ x \in V(G)\}$ and the maximum degree $\Delta(G) = \max\{d(x) :\ x \in V(G)\}$. A vertex x is an *isolated vertex* if $d(x) = 0$, an edge xy is an *isolated edge* if $d(x) = d(y) = 1$. A graph G is k-regular if $\delta(G) = \Delta(G) = k$. K_n denotes a complete graph of order n, which is an $(n-1)$-regular graph. For a subgraph H of G, we will use $\Sigma(H)$ to denote $\Sigma_{x\in H} d_H(x)$. For example, if P_3 and

C_3 are subgraphs of G, then $\Sigma(P_3) = 4$ and $\Sigma(C_3) = 6$.

Let $X \subset V(G)$ be a vertex-cut. The maximal connected subgraphs of $G - X$ are called *components*. A component is *small* if it is an isolated vertex or an isolated edge; is *large* otherwise.

In this section, we present our main theorem, which explores the close relationship between the conditional diagnosability $t_c(G)$ and the 2-extra connectivity $\kappa_2(G)$ of a regular graph G under some conditions, that is, $t_c(G) = \kappa_2(G)$. The following three lemmas play a key role in the proof of our theorem.

Lemma 2.1 [39] *Let $G = (V, E)$ be a graph, $F_1, F_2 \subseteq V(G), F_1 \neq F_2$. Then, under the comparison model, (F_1, F_2) is a distinguishable pair if and only if one of the following conditions is satisfied* (see Fig. 1).

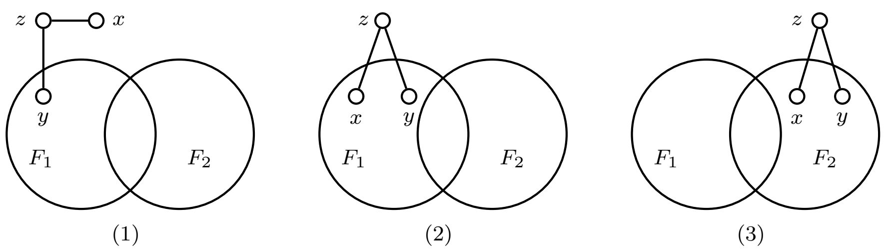

Figure 1: Illustrations of Lemma 2.1

(a) *There exists $x, z \in \overline{F_1 \cup F_2}$ and $y \in (F_1 \cup F_2) \setminus (F_1 \cap F_2)$ such that $xz, yz \in E(G)$;*

(b) *There exists $z \in \overline{F_1 \cup F_2}$ and $x, y \in F_1 \setminus F_2$ such that $xz, yz \in E(G)$;*

(c) *There exists $z \in \overline{F_1 \cup F_2}$ and $x, y \in F_2 \setminus F_1$ such that $xz, yz \in E(G)$.*

Lemma 2.2 [39] *A graph G is conditionally t-diagnosable if and only if, for any two distinct conditional fault-sets F_1 and F_2 with* $\max\{|F_1|, |F_2|\} \leqslant t$, *$(F_1, F_2)$ is a distinguishable pair.*

Lemma 2.3 [6] *Let $G = (V, E)$ be a graph with maximum degree Δ and minimum degree $\delta \geqslant 3$. If there is some integer t such that*

(a) $|V| > (\Delta + 1)(t - 1) + 4$,[1]

(b) *for any $F \subset V(G)$ with $|F| \leqslant t - 1$, $G - F$ has a large component and small components (if exist) which contain at most two vertices in total.*

then $t_c(G) \geqslant t$.

[1]This lower bound on $|V|$ given here is quite enough for the conclusion. The original article claims $|V| > (\Delta + 2)(t - 1) + 4$.

Theorem 2.4 *Let G be an n-regular R_2-graph and $t = \min\{|N(T)| : T$ is a 3-path or a 3-cycle in $G\}$. If G satisfies the following conditions*

(a) *for any $F \subset V(G)$ with $|F| \leqslant t-1$, $G-F$ has a large component and small components which contain at most two vertices in total,*

(b) *$n \geqslant 2\ell(G)+2$ if G contains no 5-cycle, and $n \geqslant 3\ell(G)+2$ otherwise,*

(c) *$|V(G)| > (n+1)(t-1)+4$,*

then $t_c(G) = t = \kappa_2(G)$.

Proof. Let $T = P_3$ or C_3 (if exists) in G such that $|N(T)| = t$. The condition (c) implies that $N(T)$ is a vertex-cut of G.

Suppose that $N(T)$ is not an R_2-vertex-cut of G. Then $G - N(T)$ contains a small component C which contains at most two vertices.

If C is an isolated vertex, say x, then x shares at most $\ell(G)$ common neighbors with any of three vertices in T. Thus, $n = |N(x) \cap N(T)| \leqslant \min\{3\ell(G), n\}$, which implies $n \leqslant 3\ell(G)$, a contradiction with the hypothesis (b) that $n \geqslant 3\ell(G)+2$. Moreover, if G contains no 5-cycle, then x shares at most $\ell(G)$ common neighbors with each of at most two vertices in T, and so $n = |N(x) \cap N(T)| \leqslant \min\{2\ell(G), n\}$, which implies $n \leqslant 2\ell(G)$, a contradiction with the hypothesis (b) that $n \geqslant 2\ell(G)+2$.

If C is an isolated edge, say xy, then at most $(n-1)$ neighbors of x are in $N(T)$. In the same discussion above, we have that $n-1 = |N(x) \cap N(T)| \leqslant \min\{3\ell(G), n-1\}$, which implies $n \leqslant 3\ell(G)+1$; and if G contains no 5-cycle, then $n-1 = |N(x) \cap N(T)| \leqslant \min\{2\ell(G), n-1\}$, which implies $n \leqslant 2\ell(G)+1$. These contradict with the condition (b). Hence, $N(T)$ is an R_2-vertex-cut of G, and so $\kappa_2(G) \leqslant |N(T)| = t$.

On the other hand, since G is an R_2-graph, there is an R_2-vertex-cut F of G such that $|F| = \kappa_2(G)$. Clearly, F is a vertex-cut of G. By the condition (a), if $|F| \leqslant t-1$, then $G-F$ certainly contains a small component C with $|V(C)| \leqslant 2$, which contradicts the assumption that F is an R_2-vertex-cut, and so $\kappa_2(G) = |F| \geqslant t$. Thus, $\kappa_2(G) = t$.

We now prove $t_c(G) = t$. The conditions (a) and (c) satisfy two conditions in Lemma 2.3, and so $t_c(G) \geqslant t$.

On the other hand, let $T = \{x, z, y\}$ with $xz, yz \in E(G)$ such that $|N(T)| = t$. By the above discussion, $N(T)$ is an R_2-vertex-cut of G. Let $F_1 = N(T) \cup \{x\}$ and $F_2 = N(T) \cup \{y\}$. Then $F_1 \neq F_2$ and $|F_1| = |F_2| = t+1$. If there is a vertex $u \in \overline{F_1}$ such that $N(u) \subseteq F_1$, then $u \notin \{y, z\}$ clearly, and so u is in $G - N[T]$. Since u is not adjacent to x, u is an isolated vertex in $G - N(T)$, which implies that $N(T)$ is not an

R_2-vertex-cut, a contradiction. Therefore, F_1 is a conditional fault-set. Similarly, F_2 is also a conditional fault-set. Note that $(F_1 \cup F_2) \setminus (F_1 \cap F_2) = \{x, y\}$, $F_1 \setminus F_2 = \{x\}$ and $F_2 \setminus F_1 = \{y\}$. It is easy to verify that F_1 and F_2 satisfy none of conditions in Lemma 2.1, and so (F_1, F_2) is an indistinguishable pair. By Lemma 2.2, G is not conditionally $(t+1)$-diagnosable, which implies $t_c(G) \leqslant t$. Thus, $t_c(G) = t$.

It follows that $t_c(G) = t = \kappa_2(G)$. The theorem follows. ∎

3 Applications to Some Well-known Networks

As applications of Theorem 2.4, in this section, we determine the conditional diagnosability and 2-extra connectivity for some well-known vertex-transitive graphs, which, due to their high symmetry, frequently appear in the literature on designs and analyses of interconnection networks, including star graphs, alternating group networks, alternating group graphs, bubble-sort graphs, (n, k)-arrangement graphs, (n, k)-star graphs, a class of Cayley graphs obtained from transposition generating trees, k-ary n-cube networks and dual-cubes as well.

3.1 Preliminary on Groups and Cayley Graphs

We first simply recall some basic concepts on groups and the definition of Cayley graphs, and introduce two classes of Cayley graphs based on the alternating group, alternating group networks and alternating group graphs.

Denote by Ω_n the group of all permutations on $I_n = \{1, \ldots, n\}$. For convenience, we use $p_1 p_2 \cdots p_n$ to denote the permutation $\binom{1\ 2\ \cdots\ n}{p_1 p_2 \cdots p_n}$. A *transposition* is a permutation that exchanges two elements and leaves the rest unaltered. A transposition that exchanges i and j is denoted by (i, j).

It is well known that any permutation can be expressed as multiplications of a series of transpositions with operation sequence from left to right. In particular, a 3-cycle (a, b, c) is always expressed as $(a, b, c) = (a, b)(a, c)$. For example, $(1, 2, 4) = (1, 2)(1, 4)$.

A permutation is called *even* if it can be expressed as a composition of even transpositions, and *odd* otherwise. There are $n!/2$ even permutations in Ω_n, which form a subgroup of Ω_n, called the *alternating group* and denoted by Γ_n, the generating set to be a set of 3-cycles.

An automorphism of a graph G is a permutation on $V(G)$ that preserves adjacency. All automorphisms of G form a group, denoted by $\mathrm{Aut}\,(G)$, and referred to as the automorphism group. A graph G is *vertex-transitive* if for any two vertices

x and y in G there is a $\sigma \in \mathrm{Aut}\,(G)$ such that $y = \sigma(x)$. A vertex-transitive graph is necessarily regular. A graph G is *edge-transitive* if for any two edges $a = xy$ and $b = uv$ of G there is a $\sigma \in Aut(G)$ such that $\{u, v\} = \{\sigma(x), \sigma(y)\}$. A graph is *symmetric* if it is vertex-transitive and edge-transitive.

For a finite group Γ with the identity e and a non-empty subset S of Γ such that $e \notin S$ and $S = S^{-1}$, define a graph G as follows.

$$V(G) = \Gamma; \quad xy \in E(G) \Leftrightarrow x^{-1}y \in S \text{ for any } x, y \in \Gamma.$$

In other words, $xy \in E(G)$ if and only if there exists $s \in S$ such that $y = xs$. Such a graph G is called the *Cayley graph* on Γ with respect to S, denoted by $C_\Gamma(S)$. A Cayley graph is $|S|$-regular, and is connected if and only if S generates Γ. Moreover, A Cayley graph is $|S|$-connected if S is a minimal generating set of Γ.

A Cayley graph is always vertex-transitive and, thus, becomes an important topological structure of interconnection networks and has attracted considerable attention in the literature [22, 31].

As examples, we recall two well-known classes of Cayley graphs on the alternating group Γ_n with respect to some S.

1. Alternating Group Networks

For $n \geqslant 3$, let $S = \{(1,2)(1,3), (1,3)(1,2),\ (1,2)(3,i) \ :\ 4 \leqslant i \leqslant n\}$, where $(1,2)(1,3)$ and $(1,3)(1,2)$ are mutually inverse, $(1,2)(3,i)$ is self-inverse for each $i = 4, \cdots, n$, and so $S = S^{-1}$. The Cayley graph $C_{\Gamma_n}(S)$ is called the *alternating group network*, proposed by Ji [27] in 1999 and denoted by AN_n, which is $(n-1)$ regular and $(n-1)$-connected. The alternating group networks AN_3 and AN_4 are shown in Fig. 2.

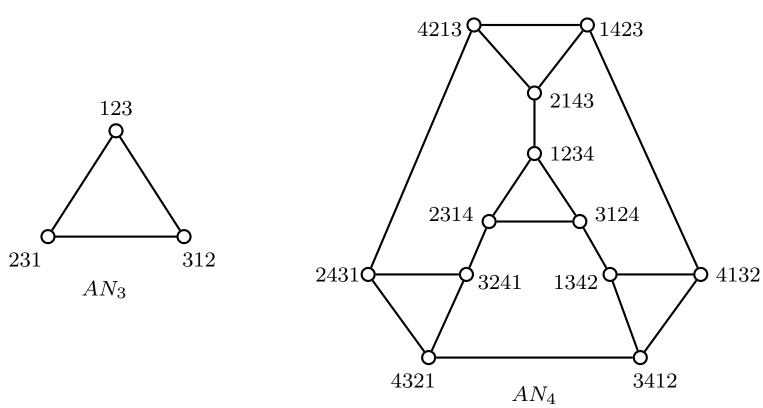

Figure 2: Alternating group networks AN_3 and AN_4

Zhou and Xiao [50] determined $t_c(AN_n) = 3n - 9$ for $n \geqslant 5$ and Zhou [47] determined $\kappa_2(AN_n) = 3n - 9$ for $n \geqslant 4$. Thus, $t_c(AN_n) = 3n - 9 = \kappa_2(AN_n)$ for $n \geqslant 5$

2. Alternating Group Graphs

For $n \geqslant 3$, let $S = \{(1,2)(1,i), (1,i)(1,2) : \ 3 \leqslant i \leqslant n\}$, where $(1,2)(1,i)$ and $(1,i)(1,2)$ are mutually inverse for each $i = 3, \cdots, n$, and so $S = S^{-1}$. The Cayley graph $C_{\Gamma_n}(S)$ is called the *alternating group graph*, proposed by Jwo *et al.* [28] in 1993 and denoted by AG_n, which is $(2n-4)$-regular and $(2n-4)$-connected. AG_3 and AG_4 are shown in Fig. 3.

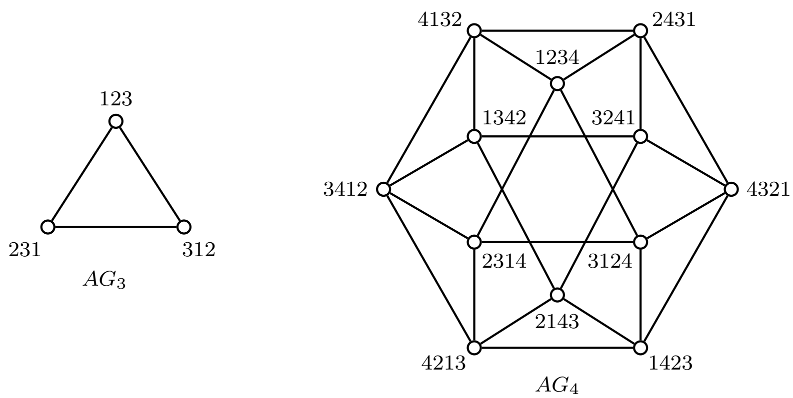

Figure 3: Alternating group graphs AG_3 and AG_4

It is known that $\kappa_2(AG_n) = 6n - 19$ for $n \geqslant 5$ determined by Lin *et al.* [34] and $t_c(AG_4) = 4$ and $t_c(AG_n) = 6n - 19$ for $n \geqslant 6$ obtained by Zhou and Xu [51], and Hao *et al.* [20], in which "$t_c(AG_n) = 6n - 18$" is a slip of the pen. Thus, $t_c(AG_n) = 6n - 19 = \kappa_2(AG_n)$ for $n \geqslant 6$.

3.2 Star Graphs

Let Ω_n be the symmetry group and $S = \{(1,i) : \ 2 \leqslant i \leqslant n\}$. The Cayley graph $C_{\Omega_n}(S)$ is called a star graph, denoted by S_n, proposed by Akers and Krishnamurthy [1] in 1989. The graphs shown in Figure 4 are S_2, S_3 and S_4.

A star graph S_n is $(n-1)$-regular and $(n-1)$-connected. Furthermore, since a transposition changes the parity of a permutation, each edge connects an odd permutation with an even permutation, and so S_n is bipartite, and contains no C_4. A star graph is not only vertex-transitive but also edge-transitive [1], and so is symmetric.

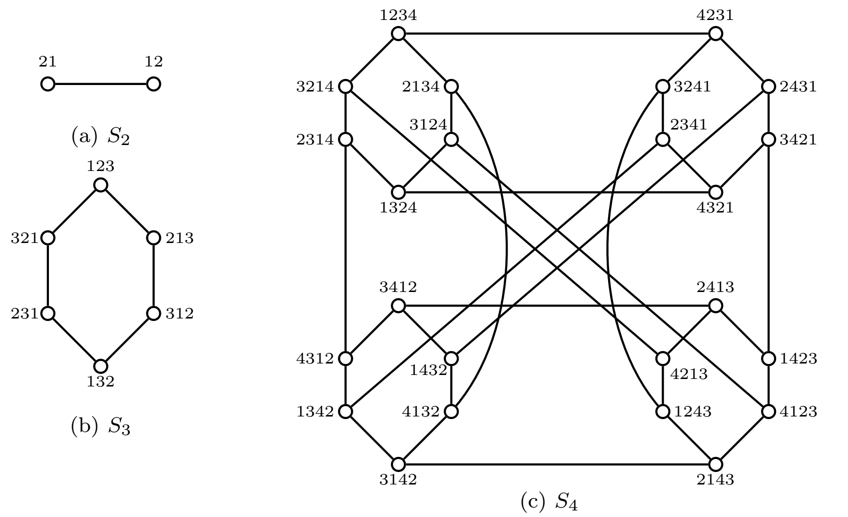

Figure 4: The star graphs S_2, S_3 and S_4

Lemma 3.1 *For any $x, y \in V(S_n)$, if $xy \notin E(S_n)$ and $N(x) \cap N(y) \neq \emptyset$, then $|N(x) \cap N(y)| = 1$.*

Since S_n is $(n-1)$-regular and contains no C_3, according to Lemma 3.1, if $P_3 = (x, y, z)$ is a 3-path, where $xz \notin E(G)$, then $|N(x) \cap N(y)| = |N(y) \cap N(z)| = 0$ and $N(x) \cap N(z) = \{y\}$, and so the number of neighbors of P_3 in S_n can be counted as follows.

$$\begin{aligned} |N(P_3)| &= d(x) + d(y) + d(z) - |N(x) \cap N(y)| - |N(y) \cap N(z)| - \Sigma(P_3) \\ &= 3(n-1) - 4 = 3n - 7. \end{aligned}$$

Since S_n is vertex-transitive, for any 3-path P_3 in S_n, we have that

$$|N(P_3)| = 3(n-1) - 4 = 3n - 7. \tag{1}$$

Lemma 3.2 (Cheng and Lipták [5]) *Let $F \subset V(S_n)$ with $|F| \leqslant 3n - 8$ and $n \geqslant 5$. If $S_n - F$ is disconnected, then it has either two components, one of which is an isolated vertex or an edge, or three components, two of which are isolated vertices.*

Lin *et al.* [33], Zhou and Xu [51] determined $t_c(S_n) = 3n - 7$ for $n \geqslant 4$. However, $\kappa_2(S_n)$ has not been determined so for. We can deduce these results by Theorem 2.4.

Theorem 3.3 $t_c(S_n) = 3n - 7 = \kappa_2(S_n)$ *for $n \geqslant 5$.*

Proof. Since S_n contains no C_3, $t = \min\{|N(T)| : T = P_3 \text{ or } C_3 \text{ in } S_n\} = |N(P_3)|$, where P_3 is any 3-path in S_n since S_n is vertex-transitive. Let $F = N(P_3)$. Then $|F| = t = 3n - 7$ by (1). It is easy to check that $|V(S_n)| - |F| - 3 = n! - 3n + 4 > 0$

for $n \geqslant 4$. Thus F is a vertex-cut of S_n. To prove the theorem, we only need to verify that S_n satisfies conditions in Theorem 2.4.

(a) If $|F| \leqslant t-1$ then, by Lemma 3.2, $S_n - F$ has a large component and small components which contain at most two vertices in total.

(b) By Lemma 3.1, $\ell(S_n) = 1$. Since S_n is $(n-1)$-regular bipartite, it contains no 5-cycle, and so $n-1 \geqslant 4 = 2\ell(S_n)+2$.

(c) When $n \geqslant 4$, it is easy to check that

$$\begin{aligned} n\,! - n(t-1) - 4 \quad &= n\,! - n(3n-8) - 4 \\ &\geqslant 4(n-1)(n-2) - 3n^2 + 8n - 4 \\ &= (n-2)^2 \\ &> 0. \end{aligned}$$

S_n satisfies all conditions in Theorem 2.4, and so $t_c(S_n) = 3n-7 = \kappa_2(S_n)$. ∎

The star graph S_n is an important topological structure of interconnection networks and has attracted considerable attention since it has been thought to be an attractive alternative to the hypercube. However, since S_n has $n\,!$ vertices, there is a large gap between $n\,!$ and $(n+1)\,!$ for expanding S_n to S_{n+1}. To relax the restriction of the numbers of vertices in S_n, the arrangement graph $A_{n,k}$ and the (n,k)-star graph $S_{n,k}$ were proposed as generalizations of the star graph S_n. In the following two sections, we discuss such two classes of graphs, respectively.

For this purpose, we need some notations. Given two positive integers n and k with $k < n$, let $P_{n,k}$ be a set of arrangements of k elements in I_n, i.e., $P_{n,k} = \{p_1p_2\dots p_k :\ p_i \in I_n, p_i \neq p_j, 1 \leqslant i \neq j \leqslant k\}$. Clearly, $|P_{n,k}| = \frac{n!}{(n-k)!}$.

3.3 Arrangement Graphs

The (n,k)-*arrangement graph*, denoted by $A_{n,k}$, was proposed by Day and Tripathi [12] in 1992. The definition of $A_{n,k}$ is as follows. $A_{n,k}$ has vertex-set $P_{n,k}$ and two vertices are adjacent if and only if they differ in exactly one position.

Figure 5 shows a $(4,2)$-arrangement graph $A_{4,2}$, which is isomorphic to AG_4 (see Fig. 3).

Since $|P_{n,k}| = \frac{n!}{(n-k)!}$ and $|S| = k(n-k)$, $A_{n,k}$ is a $k(n-k)$-regular graph with order $\frac{n\,!}{(n-k)\,!}$, and is $k(n-k)$-connected since S is a minimal generating set of Γ_n. Moreover, $A_{n,k}$ is vertex-transitive and edge-transitive (see [12]), and so $A_{n,k}$ is symmetric. Clearly, $A_{n,1} \cong K_n$ and $A_{n,n-1} \cong S_n$. Chiang and Chen [10] showed that $A_{n,n-2} \cong AG_n$. Thus, the (n,k)-arrangement graph $A_{n,k}$ is naturally regarded as a common generalization of the star graph S_n and the alternating group graph

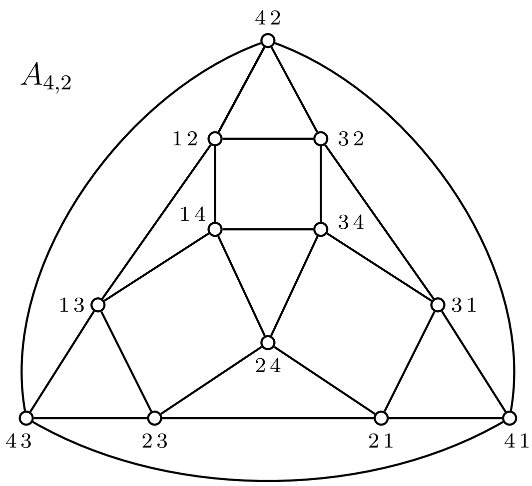

Figure 5: The structure of a $(4,2)$-arrangement graph $A_{4,2}$

AG_n. For a fixed i $(1 \leqslant i \leqslant k)$, let

$$V_i = \{p_1 \cdots p_{i-1} q_i p_{i+1} \cdots p_k : \ q_i \in I_n \setminus \{p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_k\}\}$$

Then $|V_i| = n - k + 1$. There are $|P_{n,k-1}|$ such V_i's. By definition, it is easy to see that the subgraph of $A_{n,k}$ induced by V_i is a complete graph K_{n-k+1}. In special, $K_{n-k+1} = K_n$ if $k = 1$, and $K_{n-k+1} = K_2$ if $k = n-1$.

When $n = k+1$, $A_{n,k}$ contains no 3-cycle C_3, there is a big difference in the way of dealing it with other conditions. Since $A_{n,n-1} \cong S_n$, which has been discussed in the above subsection, to avoid duplication of discussion, we may assume $n \geqslant k+2$ and $k \geqslant 2$ in the following discussion.

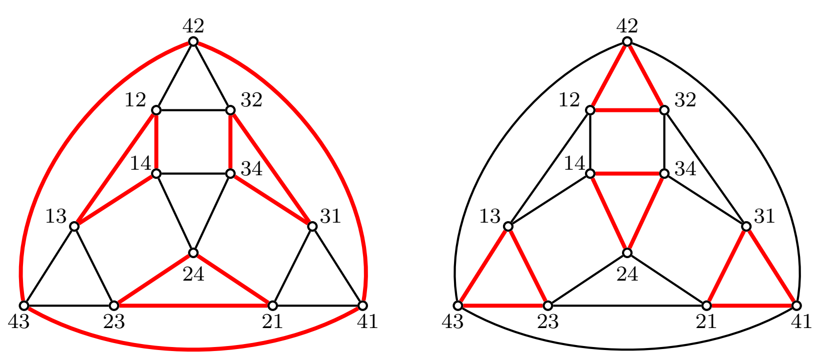

Figure 6: Two partitions of $A_{4,2}$ into 4 triangles K_3 (red edges)

Thus, when $n \geqslant k+2$ and $k \geqslant 2$, for each fixed i $(1 \leqslant i \leqslant k)$, the vertex-set of $A_{n,k}$ can be partitioned into $|P_{n,k-1}|$ subsets, each of which induces a complete graph K_{n-k+1}. For example, for $n = 4$ and $k = 2$, $|P_{4,1}| = 4$. Fig. 6 illustrates

two partitions of $V(A_{4,2})$ into 4 subsets for each $i = 1, 2$, each of which induces a complete graph K_3 (red edges). This fact and the arbitrariness of i ($1 \leqslant i \leqslant k$) show that each vertex is contained in k distinct K_{n-k+1}'s, and each edge is contained in $(n-k-1)$ distinct 3-cycles, that is, any two adjacent vertices have exactly $(n-k-1)$ common neighbors.

Furthermore, each edge of $A_{n,k}$ is contained in $(k-1)$ chordless 4-cycles when $n \geqslant k+2$ and $k \geqslant 2$. In fact, let $p\,q \in E(A_{n,k})$, if $p = p_1 \cdots p_{i-1} p_i p_{i+1} \cdots p_k$, then $q = p_1 \cdots p_{i-1} q_i p_{i+1} \cdots p_k$, where $q_i \in I_n \setminus \{p_1, \cdots, p_k\}$. For each $j \in \{1, 2, \cdots, k\}$ and $j \neq i$, let

$$\begin{array}{l} x_j = p_1 \cdots p_{i-1} q_i p_{i+1} \cdots p_{j-1} t_j p_{j+1} \cdots p_k \;\; \text{and} \\ y_j = p_1 \cdots p_{i-1} p_i p_{i+1} \cdots p_{j-1} t_j p_{j+1} \cdots p_k, \end{array}$$

where $t_j \in I_n \setminus \{p_1, \cdots, p_k, q_i\}$, such t_j certainly exists since $n \geqslant k+2$ and $k \geqslant 2$. Then, (p, q, x_j, y_j) is a chordless 4-cycle in $A_{n,k}$ for each $j \in \{1, 2, \cdots, k\}$ and $j \neq i$ (see Fig. 7).

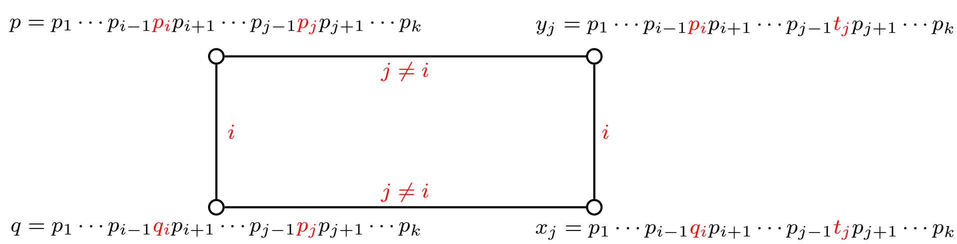

Figure 7: Construction of a chordless 4-cycle containing a given edge pq in $A_{n,k}$

According to the above discussion, we have the following result.

Lemma 3.4 *When $n \geqslant k+2$, for any $x, y \in V(A_{n,k})$, then $|N(x) \cap N(y)| = n-k-1$ if $xy \in E(A_{n,k})$; $|N(x) \cap N(y)| \leqslant 2$ if $xy \notin E(A_{n,k})$ and $N(x) \cap N(y) \neq \emptyset$; and $|N(x) \cap N(y)| = 0$ otherwise.*

Since each edge is contained in a K_{n-k+1} ($n \geqslant k+2$), for a 3-cycle $C_3 = (x, y, z)$, every vertex in $V(K_{n-k+1} - C_3)$ is a common neighbor of the three edges xy, yz, zx. In other words, when we count the number $|N(C_3)|$ of neighbors of C_3 in $A_{n,k}$, every vertex in $V(K_{n-k+1} - C_3)$ is counted three times. Thus, the number $|N(C_3)|$ of neighbors of C_3 in $A_{n,k}$ can be counted as follows.

$$\begin{aligned} |N(C_3)| &= d(x) + d(y) + d(z) - 2|V(K_{n-k+1} - C_3)| - \Sigma(C_3) \\ &= 3k(n-k) - 2(n-k-2) - 6 \\ &= (3k-2)(n-k) - 2. \end{aligned}$$

Since $A_{n,k}$ is vertex-transitive, for any 3-cycle C_3 in $A_{n,k}$, we have that

$$|N(C_3)| = (3k-2)(n-k) - 2. \tag{2}$$

Since $A_{n,k}$ contains chordless 4-cycle, say (x, y, z, u), we choose a 3-path $P_3 = (x, y, z)$. Then $xz \notin E(A_{n,k})$. Since each edge is contained in a K_{n-k+1}, $|N(x) \cap N(y)| = |N(y) \cap N(z)| = n-k-1$ and $|N(z) \cap N(x)| = |\{y, u\}| = 2$ by Lemma 3.4. Note that two edge xy and yz are in different complete graphs. Thus, the number of neighbors of P_3 in $A_{n,k}$ can be counted as follows.

$$\begin{array}{rl} |N(P_3)| & = d(x) + d(y) + d(z) - |N(x) \cap N(y)| \\ & \quad -|N(y) \cap N(z)| - |N(z) \cap N(x) \setminus \{y\}| - \Sigma(P_3) \\ & = 3k(n-k) - 2(n-k-1) - 1 - 4 \\ & = (3k-2)(n-k) - 3 \end{array}$$

Since $A_{n,k}$ is vertex-transitive, for any 3-path P_3 in $A_{n,k}$, we have that

$$|N(P_3)| = (3k-2)(n-k) - 3 \tag{3}$$

Lemma 3.5 [51] *Let F be a vertex-cut of $A_{n,k}$ with $|F| \leqslant (3k-2)(n-k) - 4$. If $n \geqslant k+2$ and $k \geqslant 4$, then $A_{n,k} - F$ contains either two components, one of which is an isolated vertex or an isolated edge, or three components, two of which are isolated vertices.*

Zhou and Xu [51] determined that for $n \geqslant k+2$ and $k \geqslant 4$, $t_c(A_{n,k}) = (3k-2)(n-k) - 3$. However, $\kappa_2(A_{n,k})$ has not been determined. We can deduce these results by Theorem 2.4.

Theorem 3.6 *$t_c(A_{n,k}) = (3k-2)(n-k) - 3 = \kappa_2(A_{n,k})$ for $n \geqslant k+2$ and $k(n-k) \geqslant 8$.*

Proof. Comparing (2) with (3), when $n \geqslant k+2$, $t = \min\{|N(T)| : T = P_3$ or C_3 in $A_{n,k}\} = |N(P_3)|$, where P_3 is any 3-path in $A_{n,k}$ since $A_{n,k}$ is vertex-transitive. Let $F = N(P_3)$. Then $|F| = t = (3k-2)(n-k) - 3$ by (3). It is easy to check that F is a vertex-cut of $A_{n,k}$. To prove the theorem, we only need to verify that $A_{n,k}$ satisfies conditions in Theorem 2.4.

(a) If $|F| \leqslant t-1$ then, by Lemma 3.5, $A_{n,k} - F$ has a large component and small components which contain at most two vertices in total.

(b) By Lemma 3.4, $\ell(A_{n,k}) = 2$, and so $k(n-k) \geqslant 8 = 3\ell(A_{n,k}) + 2$.

(c) It is not difficult to check that

$$\begin{aligned}
&|V| - [(\Delta+1)(t-1)+4] \\
&= |V| - (k(n-k)+1)((3k-2)(n-k)-4) - 4 \\
&= |V| - 3k^2(n-k)^2 + 2k(n-k)^2 + (k+2)(n-k) \\
&> |V| - 3k^2(n-k)^2 \quad (\text{for } n-k \geqslant 2) \\
&\geqslant |V| - 3(n-2)^2(n-k+1)^2 \quad (\text{for } k \leqslant n-2) \\
&= n!/(n-k)! - 3(n-2)^2(n-k+1)^2 \\
&= n(n-1)\cdots(n-k+1) - 3(n-2)^2(n-k+1)^2 \\
&> 3(n-2)^2(n-k+1)^2 - 3(n-2)^2(n-k+1)^2 \\
&= 0.
\end{aligned}$$

$A_{n,k}$ satisfies all conditions in Theorem 2.4, and so $t_c(A_{n,k}) = (3k-2)(n-k)-3 = \kappa_2(A_{n,k})$. ∎

Since $A_{n,n-2} \cong AG_n$, by Theorem 3.6, we immediately obtain the following results.

Corollary 3.7 $t_c(AG_n) = 6n-19 = \kappa_2(AG_n)$ *for* $n \geqslant 6$.

3.4 (n,k)-Star Graphs

The (n,k)-star graph $S_{n,k}$, proposed by Chiang *et al.* [9] in 1995 as another generalization of the star graph S_n, has vertex-set $P_{n,k}$, a vertex $p = p_1p_2\ldots p_i\ldots p_k$ is adjacent to a vertex

(a) $p_ip_2\cdots p_{i-1}p_1p_{i+1}\cdots p_k$, where $i \in \{2,3,\cdots,k\}$ (swap-edge).

(b) $p_1'p_2p_3\cdots p_k$, where $p_1' \in I_n \setminus \{p_i:\ i \in I_k\}$ (unswap-edge).

Figure 8 shows two (n,k)-star graphs $S_{4,3}$ and $S_{4,2}$, where $S_{4,3} \cong S_4$ and $S_{4,2} \cong AN_4$.

Since $|P_{n,k}| = \frac{n!}{(n-k)!}$ and $|S| = n-1$, $S_{n,k}$ is an $(n-1)$-regular and $(n-1)$-connected graph with order $\frac{n!}{(n-k)!}$. Moreover, $S_{n,k}$ is vertex-transitive, however, it is not edge-transitive if $n \geqslant k+2$ (see Chiang *et al.* [9]).

By definition, $S_{n,1} \cong K_n$ and $S_{n,n-1} \cong S_n$ obviously. Moreover, Cheng *et al.* [8] showed $S_{n,n-2} \cong AN_n$. Thus, the (n,k)-star graph $S_{n,k}$ is naturally regarded as a common generalization of the star graph S_n and the alternating group network AN_n.

For any $\alpha = p_2p_3\cdots p_k \in P_{n,k-1}$ $(2 \leqslant k \leqslant n)$, let

$$V_\alpha = \{p_1\alpha:\ p_1 \in I_n \setminus \{p_i:\ 2 \leqslant i \leqslant k\}.$$

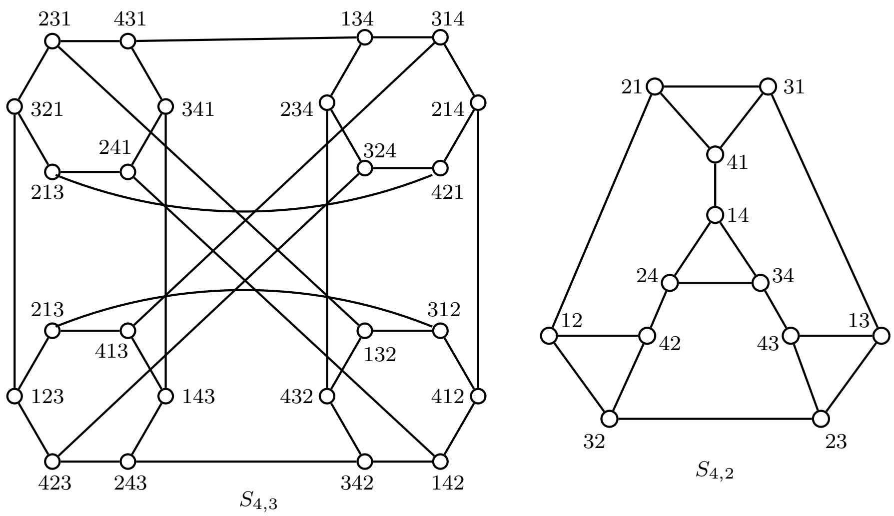

Figure 8: Two (n, k)-star graphs $S_{4,3}$ and $S_{4,2}$

By definition, it is easy to see that the subgraph of $S_{n,k}$ induced by V_α is a complete graph K_{n-k+1}. Thus, $V(S_{n,k})$ can be partitioned into $|P_{n,k-1}|$ subsets, each of which induces a complete graph K_{n-k+1} whose edges are unswap-edges. Furthermore, there is at most one swap-edge between any two complete graphs, and so $S_{n,k}$ contains neither 4-cycle nor 5-cycle.

Lemma 3.8 [32] *For any $x, y \in V(S_{n,k})$, then $|N(x) \cap N(y)| = n - k - 1$ if $xy \in E(S_{n,k})$ is an unswap-edge, $|N(x) \cap N(y)| = 1$ if $xy \notin E(S_{n,k})$ and $N(x) \cap N(y) \neq \emptyset$, and $|N(x) \cap N(y)| = 0$ otherwise.*

Since $K_{n-k+1} = K_n$ when $k = 1$ and $K_{n-k+1} = K_2$ when $k = n - 1$, like $A_{n,k}$, to avoid duplication of discussion, we may assume $n \geqslant k + 2$ and $k \geqslant 2$ in the following discussion.

For a 3-cycle $C_3 = (x, y, z)$, since it is contained in a complete graph K_{n-k+1}, every vertex in $V(K_{n-k+1} - C_3)$ is a common neighbor of the tree edges xy, yz, zx. In other words, when we count the number of neighbors of C_3 in $S_{n,k}$, every vertex in $V(K_{n-k+1} - C_3)$ is counted three times. Thus, the number of neighbors of C_3 in $S_{n,k}$ can be counted as follows.

$$\begin{aligned} |N(C_3)| &= d(x) + d(y) + d(z) - 2|V(K_{n-k+1} - C_3)| - \Sigma(C_3) \\ &= 3(n-1) - 2(n-k-2) - 6 \\ &= n + 2k - 5. \end{aligned}$$

Since $S_{n,k}$ is vertex-transitive, for any 3-cycle C_3 in $S_{n,k}$, we have that

$$|N(C_3)| = n + 2k - 5. \tag{4}$$

For a 3-path $P_3 = (x, y, z)$ with $xz \notin E(S_{n,k})$, then one of two edges xy and yz is an unswap-edge and another is a swap-edge. Without loss of generality, suppose that xy is an unswap-edge and yz is a swap-edge. Then $|N(x) \cap N(y)| = n - k - 1$, $|N(y) \cap N(z)| = 0$ and $|N(z) \cap N(x)| = |\{y\}| = 1$ by Lemma 3.8. Thus, the number of neighbors of C_3 in $A_{n,k}$ can be counted as follows.

$$\begin{aligned} |N(P_3)| &= d(x) + d(y) + d(z) - |N(x) \cap N(y)| \\ &\quad -|N(y) \cap N(z)| - |N(z) \cap N(x) \setminus \{y\}| - \Sigma(P_3) \\ &= 3(n-1) - (n-k-1) - 0 - 4 \\ &= 2n + k - 6. \end{aligned}$$

Since $S_{n,k}$ is vertex-transitive, for any 3-path P_3 in $S_{n,k}$, we have that

$$|N(P_3)| = 2n + k - 6. \tag{5}$$

Lemma 3.9 [48] *Let F be a vertex-cut of $S_{n,k}$ ($n \geqslant k+2$ and $k \geqslant 3$) with $|F| \leqslant n + 2k - 6$. Then $S_{n,k} - F$ contains either two components, one of which is an isolated vertex or an isolated edge, or three components, two of which are both isolated vertices.*

Zhou [48] determined that $t_c(S_{n,k}) = n+2k-5$ if $n \geqslant k+2$ and $k \geqslant 3$. However, $\kappa_2(S_{n,k})$ has not been determined. We can deduce these results by Theorem 2.4.

Theorem 3.10 *$t_c(S_{n,k}) = n + 2k - 5 = \kappa_2(S_{n,k})$ if $n \geqslant k+2$ and $k \geqslant 3$.*

Proof. Let $t = \min\{|N(T)| : T = P_3 \text{ or } C_3 \text{ in } S_{n,k}\}$. By Lemma 3.8, $S_{n,k}$ contains 3-cycles when $n \geqslant k+2$. Comparing (4) with (5), $t = |N(C_3)| = n+2k-5$, where C_3 is any 3-cycle in $S_{n,k}$ since $S_{n,k}$ is vertex-transitive. Let $F = N(C_3)$. Then $|F| = t$ and F is a vertex-cut of $S_{n,k}$. To prove the theorem, we only need to verify that $S_{n,k}$ satisfies conditions in Theorem 2.4.

(a) If $|F| \leqslant t-1$ then, by Lemma 3.9, $S_{n,k} - F$ has a large component and small components which contain at most two vertices in total.

(b) Since $S_{n,k}$ is $(n-1)$-regular and contains no 5-cycle C_5, by Lemma 3.8, $\ell(S_{n,k}) = 1$, and so $n - 1 \geqslant 4 = 2\ell(S_{n,k}) + 2$.

(c) It is not difficult to check that

$$\begin{aligned} |V| - [n(t-1) + 4] &= |V| - n(n + 2k - 6) - 4 \\ &\geqslant |V| - n(3n - 10) - 4 \quad (\text{for } k \leqslant n-2) \\ &\geqslant |V| - 3n(n-3) \quad (\text{for } n \geqslant 5) \\ &\geqslant n(n-1)(n-2) - 3n(n-3) \\ &> 3n(n-3) - 3n(n-3) \\ &= 0. \end{aligned}$$

$S_{n,k}$ satisfies all conditions in Theorem 2.4, and so $t_c(S_{n,k}) = n+2k-5 = \kappa_2(S_{n,k})$. The theorem follows. ∎

Since $S_{n,n-2} \cong AN_n$, by Theorem 3.10, we immediately obtain the following results.

Corollary 3.11 $t_c(AN_n) = 3n - 9 = \kappa_2(AN_n)$ *for* $n \geqslant 5$.

3.5 Transposition Graphs

Let $\mathscr{T}_n$ be a set of transpositions from Ω_n and $S \subseteq \mathscr{T}_n$. The graph T_S with vertex-set I_n and edge-set $\{ij : (i,j) \in S\}$ is called the *transposition generating graph* or simply *transposition graph*. The Cayley graph $C_{\Omega_n}(S)$ on Ω_n with respect to S has $n!$ vertices.

For example, if $S = \{(1,i) : 2 \leqslant i \leqslant n\}$, then T_S is a star $K_{1,n-1}$, the corresponding Cayley graph $C_{\Omega_n}(S)$ is a star graph S_n, proposed by Akers and Krishnamurthy [1], perhaps, this is why they called such a graph for the star graph.

Here is another example, if $S = \{(i,i+1) : 1 \leqslant i \leqslant n-1\}$, then T_S is an n-path P_n, the corresponding Cayley graph $C_{\Omega_n}(S)$ is called a bubble-sort graph B_n, proposed by Akers and Krisnamurthy [1] in 1989. This series of transpositions looks like to be along a straight line on the bubbled. Perhaps this is why Akers and Krisnamurthy called such a graph for the bubble-sort graph. Figure 9 shows the bubble-sort graphs B_2, B_3 and B_4.

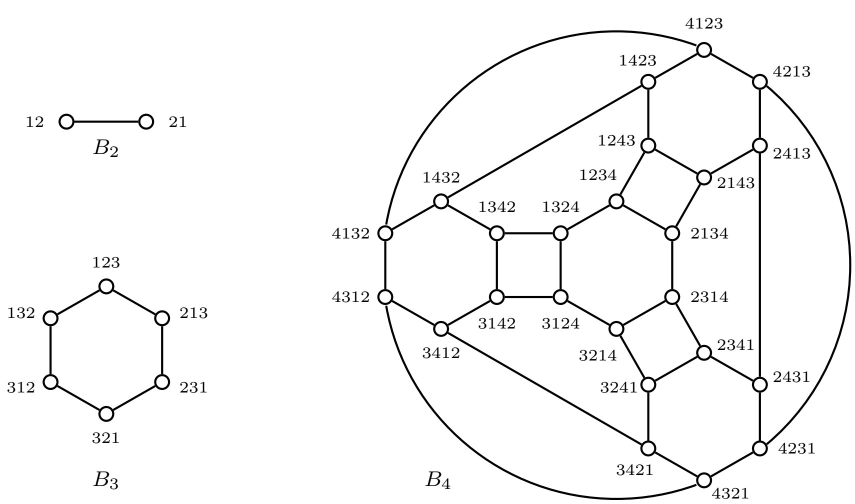

Figure 9: The bubble-sort graphs B_2, B_3 and B_4.

It is a well-known result, due to Polya (see Berge [4], p118)), that a set $S \subseteq \mathscr{T}_n$

with $|S| = (n-1)$ generates Ω_n if and only if the transposition graph T_S is a tree, called a *transposition tree*.

Thus, one is interested in such a Cayley graph $C_{\Omega_n}(S)$ obtained from a transposition generating tree T_S, denoted by $\mathscr{T}_n(S)$ shortly. The Cayley graph $\mathscr{T}_n(S)$ is a bipartite graph since a transposition changes the parity of a permutation, each edge connects an odd permutation with an even permutation.

As we have seen from the above examples, $\mathscr{T}_n(S)$ is a star graph S_n if $T_S \cong K_{1,n-1}$, and a bubble-sort graph B_n if $T_S \cong P_n$. Thus, the star graph S_n and the bubble-sort graph B_n are special cases of the Cayley graph $\mathscr{T}_n(S)$.

Since when $T_S \cong K_{1,n-1}$, $\mathscr{T}_n(S)$ is a star graph S_n. To avoid duplication of discussion, we may assume that T_S is not a star $K_{1,n-1}$ in the following discussion.

Under this assumption, when $n \geqslant 4$, Lin *et al.* [33] determined $t_c(\mathscr{T}_n(S)) = 3n-8$, Yang *et al.* [44] determined $\kappa_2(\mathscr{T}_n(S)) = 3n-8$. We can deduce these results for $n \geqslant 7$ by Theorem 2.4.

According to the recursive architecture of $\mathscr{T}_n(S)$, we easy obtain the following lemma.

Lemma 3.12 *For any $x, y \in V(\mathscr{T}_n(S))$, if $xy \notin E(\mathscr{T}_n(S))$ and $N(x) \cap N(y) \neq \emptyset$, then $|N(x) \cap N(y)| = 1$ if $\mathscr{T}_n(S) = S_n$, and $|N(x) \cap N(y)| \leqslant 2$ otherwise.*

Lemma 3.13 (Cheng and Lipták [5]) *For $n \geqslant 5$, if $T \subset V(\mathscr{T}_n(S))$ is a vertex-cut with $|T| \leqslant 3n-8$, then $\mathscr{T}_n(S) - T$ contains either two components, one of which is an isolated vertex or an isolated edge, or three components, two of which are both isolated vertices.*

Theorem 3.14 *$t_c(\mathscr{T}_n(S)) = 3n-8 = \kappa_2(\mathscr{T}_n(S))$ for $n \geqslant 7$.*

Proof. Since $\mathscr{T}_n(S)$ is a partite graph, it contains no C_3, and so $t = \min\{|N(T)| : T$ is a 3-path or a 3-cycle in $\mathscr{T}_n(S)\} = |N(P_3)|$, where P_3 is any 3-path in $\mathscr{T}_n(S)$ since $\mathscr{T}_n(S)$ is vertex-transitive. When $\mathscr{T}_n(S)$ is not a star graph, it contains C_4, and so $t = |N(P_3)| = 3(n-1) - 1 - 4 = 3n-8$. Let $F = N(P_3)$. It is easy to check that F is a vertex-cut of $\mathscr{T}_n(S)$. To prove the theorem, we only need to verify that $\mathscr{T}_n(S)$ satisfies conditions in Theorem 2.4.

(a) If $|F| \leqslant t-1$ then, by Lemma 3.13, $\mathscr{T}_n(S) - F$ has a large component and small components have at most two vertices in total.

(b) By Lemma 3.12, if $\mathscr{T}_n(S) \neq S_n$, then $\ell(\mathscr{T}_n(S)) = 2$. Since $\mathscr{T}_n(S)$ is a bipartite graph, it contains no 5-cycle C_5. It follows that $n-1 \geqslant 6 = 2\ell(A_{n,k}) + 2$.

(c) It is easy to check that $n! - [n(t-1)+4] > 0$.

$\mathscr{T}_n(S)$ satisfies all conditions in Theorem 2.4, and so $t_c(\mathscr{T}_n(S)) = 3n - 8 = \kappa_2(\mathscr{T}_n(S))$. ∎

Since when $T_S \cong P_n$ the Cayley graph $C_{\Omega_n}(S)$ is a bubble-sort graph B_n, by Theorem 3.14, we immediately obtain the following result.

Corollary 3.15 $t_c(B_n) = 3n - 8 = \kappa_2(B_n)$ *for* $n \geqslant 7$.

3.6 k-ary n-cube Networks

We first introduce the Cartesian product of graphs.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. The *Cartesian product* of G_1 and G_2 is an undirected graph, denoted by $G_1 \times G_2$, where $V(G_1 \times G_2) = V_1 \times V_2$, two distinct vertices x_1x_2 and y_1y_2, where $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$, are linked by an edge in $G_1 \times G_2$ if and only if either $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, or $x_2 = y_2$ and $x_1y_1 \in E(G_1)$.

Examples of the Cartesian product are shown in Figure 10, where $Q_1 = K_2$, $Q_i = K_2 \times Q_{i-1}$ for $i = 2, 3, 4$.

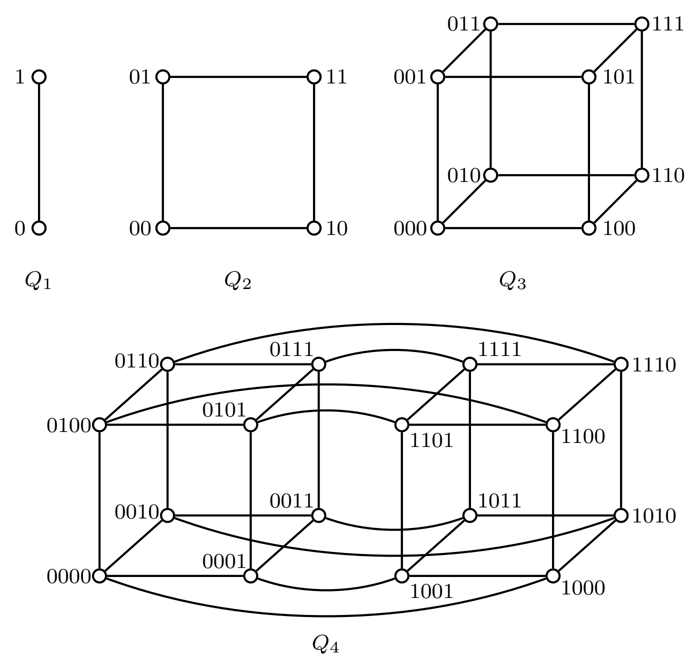

Figure 10: The hypercubes Q_n, where $Q_1 = K_2$, $Q_i = K_2 \times Q_{i-1}$ for $i = 2, 3, 4$

As an operation of graphs, the Cartesian products satisfy commutative and associative laws if we identify isomorphic graphs. Thus, we can define the Cartesian

product $G_1\times G_2\times\cdots\times G_n$. There is an edge between a vertex $x_1x_2\cdots x_n$ and another $y_1y_2\cdots y_n$ if and only if they differ exactly in the ith coordinate and $x_iy_i\in E(G_i)$.

The Cartesian product $\Gamma=\Gamma_1\times\Gamma_2\times\cdots\times\Gamma_n=(X,\circ)$ of n finite groups $\Gamma_i=(X_i,\circ_i)$ for each $i=1,2,\ldots,n$, where $X=X_1\times X_2\times\cdots\times X_n$. The operation $\circ$ is defined as follows

$$(x_1x_2\cdots x_n)\circ(y_1y_2\cdots y_n)=(x_1\circ_1 y_1)(x_2\circ_2 y_2)\cdots(x_n\circ_n y_n),$$

where $x_i,\ y_i\in X_i$ $(i=1,2,\ldots,n)$. For $x_1x_2\cdots x_n\in\Gamma$, its inverse $(x_1x_2\cdots\ x_n)^{-1}=x_1^{-1}x_2^{-1}\cdots x_n^{-1}$, the identity $e=e_1e_2\cdots e_n$, where x_i^{-1} is the inverse of x_i in Γ_i, e_i is the identity in Γ_i for each $i=1,2,\ldots,n$.

For example, consider $Z_4\times Z_2=\{00,10,20,30,01,11,21,31\}$. For any $x_1x_2,y_1y_2\in Z_4\times Z_2$, $x_1,y_1\in Z_4,\ x_2,y_2\in Z_2$, definite the operation:

$$(x_1x_2)\circ(y_1y_2)=(x_1+y_1)(\text{mod }4)(x_2+y_2)(\text{mod }2).$$

It is easy to verify that under the above operation, $Z_4\times Z_2$ forms a group, the identity is 00.

Consider the additive group $Z_k(k\geqslant 2)$ of residue classes modulo k, that is the ring group with order k, zero is the identity, the inverse of i is $k-i$. If $S=\{1\}$, then $S^{-1}=S$ for $k=2$; and $S^{-1}\neq S$ otherwise. Thus the Cayley graph $C_{Z_2}(\{1\})=K_2$, the Cayley graph $C_{Z_k}(\{1,k-1\})$ is a cycle C_k if $k\geqslant 3$.

Lemma 3.16 [42] *The Cartesian product of Cayley graphs is a Cayley graph. More precisely speaking, let $G_i=C_{\Gamma_i}(S_i)$ be a Cayley graph of a finite group Γ_i with respect to a subset S_i, then $G=G_1\times G_2\times\cdots\times G_n$ is a Cayley graph $C_\Gamma(S)$ of the group $\Gamma=\Gamma_1\times\Gamma_2\times\cdots\times\Gamma_n$ with respect to the subset*

$$S=\bigcup_{i=1}^{n}\{e_1\cdots e_{i-1}\}\times S_i\times\{e_{i+1}\cdots e_n\},$$

where e_i is the identity of Γ_i for each $i=1,2,\ldots,n$.

Let Γ be the Cartesian product of $n(\geqslant 2)$ additive groups Z_k, i.e., $\Gamma=Z_k\times Z_k\times\cdots\times Z_k$, and let

$$S=\bigcup_{i=1}^{n}\{e_1\cdots e_{i-1}\}\times S_i\times\{e_{i+1}\cdots e_n\},$$

where $e_i=0$ and $S_i=\{1,k-1\}$ for each $i=1,2,\ldots,n$. By Lemma 3.16, $C_\Gamma(S)$ is a Cayley graph. For example, let $k=2$, then

$$\begin{aligned}S&=\bigcup_{i=1}^{n}\{e_1\cdots e_{i-1}\}\times S_i\times\{e_{i+1}\cdots e_n\}\\&=\{100\cdots00,010\cdots00,\ldots,000\cdots01\},\end{aligned}$$

where $S_i = \{1\}$ for $i = 1, 2, \ldots, n$. The Cayley graph $C_\Gamma(S) = \underbrace{K_2 \times K_2 \times \cdots \times K_2}_{n}$ is the well-known hypercube Q_n.

When $k \geqslant 3$, the Cayley graph $C_\Gamma(S) = \underbrace{C_k \times C_k \times \cdots \times C_k}_{n}$ is called the *k-ary n-cube*, first studied by Dally [11] and denoted by Q_n^k (also see Xu [42]), which is an $2n$-regular graph with k^n vertices and $n\,k^n$ edges.

Lemma 3.17 [19,26] *For any $x, y \in V(Q_n^k)$, $k \geqslant 2$,*

$$|N(x) \cap N(y)| = \begin{cases} 1 & \text{if } xy \in E(Q_n^k) \text{ and } k = 3; \\ 2 & \text{if } xy \notin E(Q_n^k) \text{ and } N(x) \cap N(y) \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.18 [18,19,25] *Let F be a vertex-cut of Q_n^k ($n \geqslant 5$) with*

$$|F| \leqslant \begin{cases} 6n-6 & \text{if } k \geqslant 4; \\ 6n-8 & \text{if } k = 3; \\ 3n-6 & \text{if } k = 2. \end{cases}$$

Then $Q_n^k - F$ has a large component and small components have at most two vertices in total.

Xu *et al.* [43] determined $\kappa_2(Q_n^2) = 3n-5$ for $n \geqslant 4$. Zhao and Jin [46] determined $\kappa_2(Q_n^3) = 6n-7$ for $n \geqslant 3$. Hsieh *et al.* [23] determined $\kappa_2(Q_n^k) = 6n-5$ for $k \geqslant 4$ and $n \geqslant 5$. Hsu *et al.* [25] proved $t_c(Q_n^2) = 3n-5$ for $n \geqslant 5$. By Theorem 2.4, we immediately obtain the following result which contains the above results.

Theorem 3.19 *For $n \geqslant 8$ if $k = 5$ and $n \geqslant 6$ otherwise, $t_c(Q_n^k) = t = \kappa_2(Q_n^k)$, where*

$$t = \begin{cases} 6n-5 & \text{if } k \geqslant 4; \\ 6n-7 & \text{if } k = 3; \\ 3n-5 & \text{if } k = 2. \end{cases}$$

Proof. Note that Q_n^k is n-regular for $k = 2$, and $2n$-regular for $k \geqslant 3$, and Q_n^k contains C_3 if and only if $k = 3$ and contains C_5 if and only if $k = 5$. By Lemma 3.17, it is easy to verify that $t = \min\{|N(T)| : T = P_3 \text{ or } C_3 \text{ in } Q_n^k\} = |N(P_3)|$, where P_3 is any 3-path in Q_n^k since Q_n^k is vertex-transitive.

Let $F = N(P_3)$. Then F is a vertex-cut of Q_n^k and $|F| = t$. To prove the theorem, we only need to verify that Q_n^k satisfies conditions in Theorem 2.4.

(a) If $|F| \leqslant t-1$, then by Lemma 3.18, $Q_n^k - H$ has a large component and small components has at most two vertices in total.

(b) By Lemma 3.17, $n \geqslant 3\ell(Q_n^k)+2=8$ if $k=5$, and $n \geqslant 2\ell(Q_n^k)+2=6$ otherwise.

(c) For $n \geqslant 8$, it is easy to verify that

$$|V(Q_n^k)|-(\Delta+1)(t-1)-4=\begin{cases} 2^n-(n+1)(3n-6)-4>0 & \text{if } k=2; \\ 3^n-(2n+1)(6n-8)-4>0 & \text{if } k=3; \\ k^n-(2n+1)(6n-6)-4>0 & \text{if } k \geqslant 4. \end{cases}$$

Thus, Q_n^k satisfies all conditions in Theorem 2.4, and so $t_c(Q_n^k)=t=\kappa_2(Q_n^k)$ for $n \geqslant 8$ if $k=5$ and $n \geqslant 6$ otherwise. ∎

Corollary 3.20 *$t_c(Q_n^2)=3n-5=\kappa_2(Q_n^2)$ and $t_c(Q_n^3)=6n-7=\kappa_2(Q_n^3)$ for $n \geqslant 6$*

3.7 Dual-Cubes

A dual-cube DC_n, proposed by Li and Peng [37], consists of 2^{2n+1} vertices, and each vertex is labeled with a unique $(2n+1)$- bits binary string and has $n+1$ neighbors. There is a link between two nodes $u=u_{2n}u_{2n-1}\ldots u_0$ and $v=v_{2n}v_{2n-1}\ldots v_0$ if and only if u and v differ exactly in one bit position i under the the following conditions:

(a) if $0 \leq i \leq n-1$, then $u_{2n}=v_{2n}=0$; and

(b) if $n \leq i \leq 2n-1$, then $u_{2n}=v_{2n}=1$.

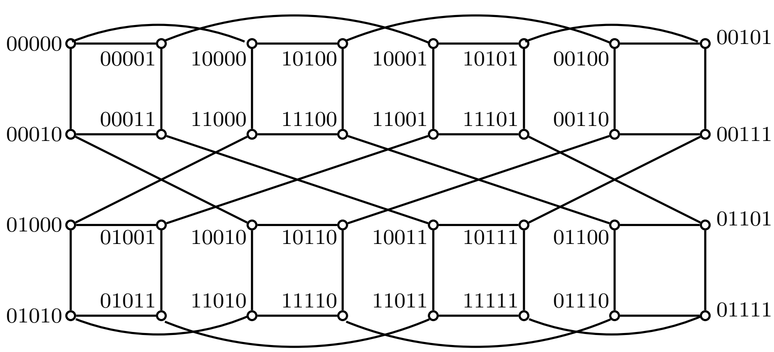

Figure 11: The dual-cube DC_2

Figure 11 shows the bubble-sort graphs DC_2. A dual-cube DC_n is an $(n+1)$-regular bipartite graph of order 2^{2n+1}. Moreover, Zhou *et al.* [49] showed that DC_n is a Cayley graph, and so DC_n is vertex-transitive.

Lemma 3.21 (Zhou *et al.* [49]) *For any $x, y \in V(DC_n)$, if $xy \notin E(DC_n)$ and $N(x)\cap N(y)\neq\emptyset$, then $|N(x)\cap N(y)| \leqslant 2$.*

Since DC_n is an $(n+1)$-regular bipartite graph, and so it contains no C_3, according to Lemma 3.21, if $P_3 = (x, y, z)$ is a 3-path, where $xz \notin E(G)$, then $|N(x) \cap N(y)| = |N(y) \cap N(z)| = 0$ and $|N(x) \cap N(z)| \leqslant 2$, and so the number of neighbors of P_3 in DC_n can be counted as follows.

$$\begin{aligned} |N(P_3)| &= d(x) + d(y) + d(z) - |N(x) \cap N(z)| - \Sigma(P_3) \\ &= 3(n+1) - |N(x) \cap N(z)| - 4 \\ &= \begin{cases} 3n-1 & \text{if} |N(x) \cap N(z)| = 1; \\ 3n-2 & \text{if} |N(x) \cap N(z)| = 2. \end{cases} \end{aligned}$$

Since DC_n is vertex-transitive, for any 3-path P_3 in DC_n that $|N(P_3)|$ is the smallest, we have that

$$|N(P_3)| = 3n - 2. \tag{6}$$

Lemma 3.22 (Zhou *et al.* [49]) *Let $F \subset V(DC_n)$ with $|F| \leqslant 3n-3$ and $n \geqslant 3$. If $DC_n - F$ is disconnected, then it has either two components, one of which is an isolated vertex or an edge, or three components, two of which are isolated vertices.*

Zhou *et al.* [49] determined $\kappa_2(DC_n) = 3n-2$ and $t_c(DC_n) = 3n-2$ for $n \geq 3$, dependently. By Theorem 2.4, we immediately obtain the following result which contains the above results.

Theorem 3.23 $t_c(DC_n) = 3n - 2 = \kappa_2(DC_n)$ *for* $n \geqslant 5$.

Proof. Since DC_n contains no C_3, $t = \min\{|N(T)| : T = P_3 \text{ or } C_3 \text{ in } DC_n\} = |N(P_3)|$, where P_3 is any 3-path in DC_n since DC_n is vertex-transitive. Let $F = N(P_3)$. Then $|F| = t = 3n-2$ by (6). It is easy to check that F is a vertex-cut of DC_n. To prove the theorem, we only need to verify that DC_n satisfies conditions in Theorem 2.4.

(a) If $|F| \leqslant t-1$ then, by Lemma 3.22, $DC_n - F$ has a large component and small components which contain at most two vertices in total.

(b) By Lemma 3.21, $\ell(DC_n) = 2$. Since DC_n is $(n+1)$-regular bipartite, it contains no 5-cycle, and so $n+1 \geqslant 6 = 2\ell(DC_n) + 2$.

(c) It is easy to check that $2^{2n+1} - (n+2)(t-1) - 4 = 2^{2n+1} - (n+2)(3n-3) - 4 > 0$ for $n \geqslant 5$.

DC_n satisfies all conditions in Theorem 2.4, and so $t_c(DC_n) = 3n-2 = \kappa_2(DC_n)$. The theorem follows. ∎

4 Conclusions

The conditional diagnosability $t_c(G)$ under the comparison model and the 2-extra connectivity $\kappa_2(G)$ are two important parameters to measure ability of diagnosing faulty processors and fault-tolerance in a multiprocessor system G with the presence of failing processors. Although these two parameters have attracted considerable attention and determined for many classes of well-known graphs in recent years, but are obtained independently. This paper establishes the close relationship between these two parameters by proving $t_c(G) = \kappa_2(G)$ for a regular graph G with some acceptable conditions. As applications, the conditional diagnosability and the 2-extra connectivity are determined for some well-known classes of vertex-transitive graphs such as star graphs, (n, k)-star graphs, (n, k)-arrangement graphs, Cayley graphs obtained from transposition generating trees, k-ary n-cube networks and dual-cubes. Furthermore, many known results about these networks are obtained directly.

Under the comparison diagnosis model, the diagnosability and the 1-extra connectivity should have some relationships. On the other hand, in addition to the comparison diagnosis model, there are several other diagnosis models such as the PMC model. Under the PMC model, what is the relationship between the diagnosability or the conditional diagnosability and the h-extra connectivity for some h? These will be explored in future.

References

[1] S. B. Akers and B. Krisnamurthy, A group theoretic model for symmetric interconnection networks. IEEE Transactions on Computers, 38(4)(1989) 555–566.

[2] T. Araki, Y. Shibata, Diagnosability of networks by the cartesian product, IEICE Transactions on Fundamentals, E83-A (3)(2000) 465–470.

[3] T. Araki, Y. Shibata, Diagnosability of butterfly networks under the comparison approach, IEICE Transactions on Fundamentals E85-A (5)(2002) 1152–1160.

[4] C. Berge, Principles of Combinatorics, Academic Press, New York, 1971.

[5] E. Cheng, L. Lipták, Structural properties of cayley graphs generated by transposition trees, Congressus Numerantium, 180(2006) 81–96.

[6] E. Cheng, L. Lipták, K. Qiu, Z. Shen, On deriving conditional diagnosability of interconnection networks, Information Processing Letters, 112(17-18)(2012) 674–677.

[7] E. Cheng, L. Lipták, and F. Sala, Linearly many faults in 2-treegenerated networks, Networks, 55(2)(2010) 90–98.

[8] E. Cheng, K. Qiu, Z. Shen, A note on the alternating group network, The Journal of Supercomputing, 59(1)(2012) 246–248.

[9] W. K. Chiang and R. J. Chen, The (n, k)-star graphs: A generalized star graph, Information Processing Letters, 56(1995) 259–264.

[10] W.K. Chiang and R.J. Chen, On the arrangement graph, Information Processing Letters, 66(4)(1998) 215–219.

[11] W.J. Dally, Performance analysis of k-ary n-cube interconnection networks. IEEE Transaction on Computers, 39(6)(1990) 775–785.

[12] K. Day and A. Tripathi, Arrangement graphs: a class of generalized star graphs, Information Processing Letters, 42(5)(1992) 235–241.

[13] E.P. Duarte Jr., R.P. Ziwich, L.C.P. Albini, A Survey of Comparison-Based System-Level Diagnosis, ACM Computing Surveys, ISSN 0360-0300, 43(3)(2011) 1–56.

[14] J.X. Fan, Diagnosability of crossed cubes under the two strategies, Chinese Journal of Computers, 21(5)(1998) 456–462.

[15] J.X. Fan, Diagnosability of the Mobius cubes, IEEE Transactions on Parallel and Distributed Systems, 9(9)(1998) 923–928.

[16] J.X Fan, Diagnosability of crossed cubes under the comparison diagnosis model, IEEE Transactions on Parallel and Distributed Systems, 13(7)(2002) 687–692.

[17] J. Fàbrega, M.A. Fiol, On the extraconnectivity of graphs, Discrete Math., 155(1-3)(1996), 49-57.

[18] M.-M. Gu, R.-X. Hao and J.-B. Liu, 3-extra connectivity of k-ary n-cube networks. arXiv: 13094961V1, 19 Sep. 2013.

[19] M.-M. Gu, R.-X. Hao, 3-extra connectivity of 3-ary n-cube networks, Department of Mathematics, Information Processing Letters, 114(2014) 486–491.

[20] R.-X. Hao, Y.-Q. Feng, J. -X. Zhou, Conditional diagnosability of alternating group graphs, IEEE Transactions on Computers, 62(4)(2013) 827-831.

[21] R.-X. Hao, J.-X. Zhou , Characterize a kind of fault tolerance of alternating group network (in Chinese), Acta Mathematica Sinica, Chinese series, 55(6)(2012) 1055–1066.

[22] M.C. Heydemann, B. Ducourthial, Cayley graphs and interconnection networks, in: G. Hahn, G. Sabidussi (Eds.), Graph Symmetry (Montreal, PQ, 1996), NATO Advanced Science Institutes Series C, in: Mathematica and Physical Sciences, vol. 497, Kluwer Academic Publishers, Dordrecht, 1997, pp. 167–224.

[23] S.Y. Hsieh, Y.-H. Chang, Extra connectivity of k-ary n-cube networks, Theoretical Computer Science, 443(2012) 63–69.

[24] G.-H. Hsu, C.-F. Chiang and Jimmy J.-M. Tan, Comparison-based Conditional diagnosability on the class of hypercube-like networks, Journal of Interconnection Networks, 11(3-4)(2010), 143–156.

[25] G.-H. Hsu, C.-F. Chiang, L.-M. Shih, L.-H. Hsu and Jimmy J.-M. Tan, Conditional diagnosability of hypercubes under the comparison diagnosis model, Journal of Systems Architecture, 55(2009) 140-146.

[26] S.-Y. Hsieh, T.J. Lin, H.L. Huang, Panconnectivity and edge-pancyclicity of 3-ary n-cubes, The Journal of Supercomputing, 42(2007) 233–255.

[27] Y.-H. Ji, A new class of Cayley networks based on the alternating groups (in Chinese). Appl Math A, J Chin Univ, 14(2)(1999) 235–239. An English abstrct: A class of Cayley networks based on the alternating groups, Advances in Mathematics, Chinese, 27(4)(1998) 361–362.

[28] J.S. Jwo, S. Lakshmivarahan, and S.K. Dhall, A New Class of Interconnection Networks Based on the Alternating Group, Networks, 23(1993) 315–326.

[29] A. Kavianpour, K.H. Kim, Diagnosability of hypercube under the pessimistic one-step diagnosis strategy, IEEE Transactions on Computers 40(2)(1991) 232–237.

[30] P.-L. Lai, J.J.M. Tan, C.-P. Chang, L.-H. Hsu, Conditional diagnosability measures for large multiprocessor systems, IEEE Transactions on Computers, 54(2)(2005) 165–175.

[31] S. Lakshmivarahan, J.-S. Jwo and S.K. Dhall, Symmetry in interconnection networks based on Cayley graphs of permutation groups: A survey, Parallel Computing, 19(1993) 361–407.

[32] X.-J. Li and J.-M. Xu, Fault-tolerance of (n, k)-star networks. Applied Mathematics and Computation, 248(2014) 525–530.

[33] C.-K. Lin, J.J.M. Tan, L.-H. Hsu, E. Cheng, L. Lipt ak, Conditional diagnosability of Cayley graphs generalized by transposition tree under the comparison model, Journal of Interconnection networks, 9(2008) 83–97.

[34] L. Lin, S. Zhou, L. Xu, D. Wang, The extra connectivity and conditional diagnosabiltiy of alternating group networks, IEEE Transactions on Parallel and Distributed Systems, 26(8)(2015) 2352–2362.

[35] J. Maeng, M. Malek, A comparison connection assignment for self-diagnosis of multiprocessors systems, in: Proceedings of the 11th International Symposium on Fault-Tolerant Computing, New York, ACM Press, (1981) 173–175.

[36] M. Malek, A comparison connection assignment for diagnosis of multiprocessors systems, in: Proceedings of the 7th annual symposium on Computer Architecture, New York: ACM Press, (1980) 31–36.

[37] Y. Li, S. Peng, Dual-cubes: a new interconnection network for high-performance computer clusters, Proceedings of the 2000 international computer symposium, workshop on computer architecture, 2000, 51–57.

[38] S.L. Scott, J.R. Goodman, The impact of pipelined channel on k-ary n-cube networks, IEEE Transactions on Parallel and Distributed Systems, 5(1)(1994) 2–16.

[39] A. Sengupta, A. Dahbura, On self-diagnosable multiprocessor systems: diagnosis by the comparison approach, IEEE Transactions on Computers, 41(11)(1992) 1386–1396.

[40] D. Wang, Diagnosability of enhanced hypercubes, IEEE Transactions on Computers 43(9)(1994) 1054–1061.

[41] J.-M. Xu, Theory and Application of Graphs. Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.

[42] J.-M. Xu, Combination of Network Theory. Science Press, Beijing. 2013.

[43] Xu, J.-M., Zhu, Q., Hou, X.-M., and Zhou, T., On restricted connectivity and extra connectivity of hypercubes and folded hypercubes. Journal of Shanghai Jiaotong University (Science), E-10(2)(2005) 208–212.

[44] W.-H. Yang, C.-H. Li and J.-X. Meng, Conditional connectivity of Cayley graphs generated by transposition trees, Information Processing Letters, 110(23)(2010) 1027–1030.

[45] Z. Zhang, W. Xiong, W.H. Yang, A kind of conditional fault tolerance of alternating group graphs, Information Processing Letters, 110(2010) 998–1002.

[46] Y.-Q. Zhao and X.-H. Jin, Second-extra connectivity of 3-ary n-cube networks (in Chinese), J. Computer Applications, 33(4)(2013) 1036–1038.

[47] S.-M. Zhou, The study of fault tolerance on alternating group networks, in: Biomedical Engineering and Informatics, 2009. BMEI '09. 2nd International Conference on, Issue Date: 17-19 Oct. 2009, DOI: 10.1109/BMEI.2009.5305876.

[48] S.-M. Zhou, The conditional fault diagnosability of (n, k)-star graphs, Applied Mathematics and Computation, 218(2012) 9742–9749.

[49] Shuming Zhou and Lanxiang Chen, and J.-M. Xu, Conditional fault diagnosability of dual-cubes. International Journal of Foundations of Computer Science, 23(8)(2012) 1729–1749.

[50] S.-M. Zhou, W.-J. Xiao, Conditional diagnosability of alternating group networks, Information Processing Letters, 110(2010) 403–409.

[51] S.-M. Zhou and J.-M. Xu, Fault diagnosability of arrangement graphs. Information Sciences, 246(10)(2013) 177–190.